Midterm 2: Math 20D Midterm Exam 05/24/2023
Instructor: Finley McGlade

You have 50 minutes.

You are permitted the use of a **scientific** calculator and a double sided page of handwritten notes.

UNLESS THE QUESTION SPECIFICALLY SAYS OTHERWISE YOU MUST SHOW ALL YOUR WORKING.

Name ______________________________

PID ______________________________

"I have adhered to UCSD policies on academic integrity while completing this examination."

Signature ______________________________

The exams consists of 8 pages (including the cover page) with four questions. The maximum possible score is **30+30+30+20=110 points**. The first thing you should do when writing time begins is CHECK to make sure you have all 8 of the pages.

Good luck!

Scratch Working Page

Problem 1. *(30 points) Answer the question below by either writing TRUE or FALSE. No further justification is required. In parts (a) and (b) we make reference to the differential equation*

$$ty'' - y' + (1-t)y = te^t \tag{1}$$

(a) *(6 points) True or False. The function $y(t) = e^t$ defines a particular solution to equation (1).*

(b) *(6 points) True or False. If $v(t)$ is a function that satisfies $v'(t) = te^{-2t}$ then $y(t) = v(t)e^t$ solves the homogeneous equation associated to (1).*

(c) *(6 points) True or False. If a pair of functions $y_1, y_2\colon \mathbb{R} \to \mathbb{R}$ satisfy $\mathrm{Wr}[y_1, y_2](t) = t^3$ then the functions $y_1(t)$ and $y_2(t)$ are linearly independent over $\mathbb{R}$.*

(d) *(6 points) True or False. If $f\colon [0,\infty) \to \mathbb{R}$ satisfies $\mathcal{L}\{f(t)\}(s) = \frac{s}{s^2-4}$ then*

$$\mathcal{L}\{e^{2t}f(t)\}(s) = \frac{1}{2}\left(\frac{1}{s-4} + \frac{1}{s}\right).$$

(e) *(6 points) True or False.*

$$\mathcal{L}^{-1}\left\{\frac{e^{-3s} - 2se^{-s}}{s^2+4}\right\}(t) = \sin(t-3)\cos(t-3) - 2\cos(t-1).$$

Problem 2. *(30 points)*

(a) *(15 points) Using the method of variation of parameters, find a particular solution to the differential equation*

$$y''(t) + 4y(t) = \frac{1}{\sin(2t)}.$$

(b) *(15 points) Solve the equation in (a) for the initial conditions* $y(\pi/4) = y'(\pi/4) = 0$.

Problem 3. *(30 points)*

(a) (15 points) Calculate the inverse Laplace transform

$$\mathcal{L}^{-1}\left\{\frac{18}{(s-2)(s^2+2s+10)}\right\}(t).$$

(b) (15 points) Using the method of Laplace transform, solve the initial value problem

$$y''(t)+2y'(t)+10y(t)=18e^{2t}, \qquad y(0)=0, \quad y'(0)=3.$$

Problem 4. *(20 points) All that is known concerning a mysterious differential equation*

$$y''(t) + p(t)y'(t) + q(t)y(t) = g(t)$$

is that $e^t + \cos^2(t)$, $e^t - \sin(t)^2$, *and* $e^t + \sin^2(t)$ *are solutions.*

(a) *(10 points) Show that* $y_1(t) = 1$ *and* $y_2(t) = \cos(2t)$ *give a pair of linearly independent solutions to the associated homogeneous equation*

$$y''(t) + p(t)y'(t) + q(t)y(t) = 0.$$

`Hint: The identity` $\cos(2t) = \cos^2(t) - \sin^2(t)$ `may prove useful here.`

(b) *(10 points) Calculate the function* $p(t)$.

Scratch Working Page

A TABLE OF LAPLACE TRANSFORMS

$f(t)$	$F(s) = \mathscr{L}\{f\}(s)$	$f(t)$	$F(s) = \mathscr{L}\{f\}(s)$
1. $f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$	**20.** $\frac{1}{\sqrt{t}}$	$\frac{\sqrt{\pi}}{\sqrt{s}}$
2. $e^{at}f(t)$	$F(s-a)$	**21.** $\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{3/2}}$
3. $f'(t)$	$sF(s) - f(0)$	**22.** $t^{n-(1/2)}, \quad n = 1, 2, \ldots$	$\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)\sqrt{\pi}}{2^n s^{n+(1/2)}}$
4. $f^{(n)}(t)$	$s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0)$	**23.** $t^r, \quad r > -1$	$\frac{\Gamma(r+1)}{s^{r+1}}$
	$- \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$	**24.** $\sin bt$	$\frac{b}{s^2+b^2}$
5. $t^n f(t)$	$(-1)^n F^{(n)}(s)$	**25.** $\cos bt$	$\frac{s}{s^2+b^2}$
6. $\frac{1}{t}f(t)$	$\int_s^\infty F(u)\,du$	**26.** $e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$
7. $\int_0^t f(v)\,dv$	$\frac{F(s)}{s}$	**27.** $e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
8. $(f * g)(t)$	$F(s)G(s)$	**28.** $\sinh bt$	$\frac{b}{s^2-b^2}$
9. $f(t+T) = f(t)$	$\frac{\int_0^T e^{-st}f(t)\,dt}{1-e^{-sT}}$	**29.** $\cosh bt$	$\frac{s}{s^2-b^2}$
10. $f(t-a)u(t-a),\ a \geq 0$	$e^{-as}F(s)$	**30.** $\sin bt - bt\cos bt$	$\frac{2b^3}{(s^2+b^2)^2}$
11. $g(t)u(t-a), \quad a \geq 0$	$e^{-as}\mathscr{L}\{g(t+a)\}(s)$	**31.** $t\sin bt$	$\frac{2bs}{(s^2+b^2)^2}$
12. $u(t-a), \quad a \geq 0$	$\frac{e^{-as}}{s}$	**32.** $\sin bt + bt\cos bt$	$\frac{2bs^2}{(s^2+b^2)^2}$
13. $\prod_{a,b}(t), \quad 0 < a < b$	$\frac{e^{-sa} - e^{-sb}}{s}$	**33.** $t\cos bt$	$\frac{s^2-b^2}{(s^2+b^2)^2}$
14. $\delta(t-a), \quad a \geq 0$	e^{-as}	**34.** $\sin bt\cosh bt - \cos bt\sinh bt$	$\frac{4b^3}{s^4+4b^4}$
15. e^{at}	$\frac{1}{s-a}$	**35.** $\sin bt\sinh bt$	$\frac{2b^2 s}{s^4+4b^4}$
16. $t^n, \quad n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}}$	**36.** $\sinh bt - \sin bt$	$\frac{2b^3}{s^4-b^4}$
17. $e^{at}t^n, \quad n = 1, 2, \ldots$	$\frac{n!}{(s-a)^{n+1}}$	**37.** $\cosh bt - \cos bt$	$\frac{2b^2 s}{s^4-b^4}$
18. $e^{at} - e^{bt}$	$\frac{(a-b)}{(s-a)(s-b)}$	**38.** $J_\nu(bt), \nu > -1$	$\frac{\left(\sqrt{s^2+b^2}-s\right)^\nu}{b^\nu\sqrt{s^2+b^2}}$
19. $ae^{at} - be^{bt}$	$\frac{(a-b)s}{(s-a)(s-b)}$		